Intrinsically p-biharmonic maps

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Abstract

For a compact Riemannian manifold N, a domain $\Omega \subset \mathbb{R}^m$ and for $p \in (1, \infty)$, we introduce an intrinsic version E_p of the p-biharmonic energy functional for maps $u : \Omega \to N$. This requires finding a definition for the intrinsic Hessian of maps $u : \Omega \to N$ whose first derivatives are merely p-integrable. We prove, by means of the direct method, existence of minimizers of E_p within the corresponding intrinsic Sobolev space, and we derive a monotonicity formula. Finally, we also consider more general functionals defined in terms of polyconvex functions.

1 Introduction

Let $m \geq 4$, let $\Omega \subset \mathbb{R}^m$ be a bounded Lipschitz domain and let N be a smooth, compact Riemannian manifold without boundary, which for simplicity we assume to be embedded in some Euclidean space $\mathbb{R}^n$. We denote by A the second fundamental form of N.

Let $p \in (1, \infty)$. In this paper we introduce and analyze some variational problems for maps $u : \Omega \to N$, related to the harmonic map problem. Harmonic maps are the critical points of the Dirichlet energy

$$\int_\Omega |\nabla u|^2.$$

Among the generalizations studied in the literature are p-harmonic maps (see, e.g., [3]), coming from the functional

$$\int_\Omega |\nabla u|^p,$$

and biharmonic maps (see [4]), involving the Hessian (or Laplacian) of u rather than the gradient. The Hessian may be defined with respect to the ambient space $\mathbb{R}^n$ (i.e., extrinsically) or purely in terms of the geometry of N (intrinsically). The latter is the point of view that we take in this paper. We write Ddu for the (intrinsic) Hessian of u; a precise definition will follow. With this notation, the most important of the functionals that we study is

$$E_p(u) = \int_\Omega |Ddu|^p,$$

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although we consider other functionals as well. Our main results concern the existence of minimizers under suitable boundary conditions. The proofs rely on the direct method in the calculus of variations.

In order to apply the direct method, one has to find a suitable function space which is closed under the weak compactness enjoyed by sublevel sets of E_p. In [6], the analogous problem was studied for the case $p = 2$. There the space $H^2_N(\Omega)$ was found to be natural in this context. It is defined as the subset of those $u \in W^{1,2}(\Omega, N)$ for which the distributions

$$(Ddu)_{\alpha\beta} := \partial_\alpha \partial_\beta u + A(u)(\partial_\alpha u, \partial_\beta u) \tag{1}$$

belong to $L^2(\Omega)$ for all $\alpha, \beta = 1, ..., m$. This definition makes sense for $u \in H^1(\Omega, N)$ because then the right-hand side of (1) is a distribution in $H^{-1} + L^1$. In [6] minimizers of E_2 were constructed in the space $H^2_N(\Omega)$.
Here we follow a similar route. Indeed, for $p > 2$ it is not difficult to generalize the arguments, and thus we mostly focus on the case $p < 2$. However, if $p \in (1, 2)$ and if merely $u \in W^{1,p}(\Omega, \mathbb{R}^n)$, then the second term in (1) is no longer well-defined as a distribution. Even the integrability $u \in W^{1,\frac{mp}{m-p}}$, which one might expect from some intrinsic Sobolev inequality, is not enough to give a meaning to the right-hand side of (1) for general domain dimensions m. It is therefore not even clear how to define the intrinsic Hessian for maps $u \in W^{1,p}(\Omega, N)$, and hence it is not clear which space should replace the intrinsic space H^2_N from [6].
We will nevertheless find a natural definition of the intrinsic Hessian for maps $u \in W^{1,p}(\Omega, N)$ and we will introduce the natural space $W^{2,p}_N(\Omega)$ consisting of those maps $u \in W^{1,p}(\Omega, N)$ which have finite E_p-energy. This requires a careful analysis of maps with p-integrable intrinsic Hessian. It is based on some fine properties of Sobolev maps, and it leads to some technical results that are new even in the case $p = 2$; moreover, it leads to simpler proofs of intrinsic Sobolev-type inequalities, too. In particular, we show that maps with p-integrable intrinsic Hessian have first derivatives which are absolutely continuous along almost every coordinate line. Hence maps in $W^{2,p}_N(\Omega)$ are twice differentiable almost everywhere in Ω.
Apart from the construction of minimizers of the functionals E_p, we derive two other results. First, we show that the variational problem gives rise to a monotonicity formula, similar to a well-known formula for harmonic maps [7] with known generalizations for p-harmonic and biharmonic maps. These monotonicity formulas play an important role in the regularity theories for the problems in supercritical dimensions. We do not derive regularity here, but it is interesting to see that a similar tool is still available, although we will also see that it is more complicated for our problem. Second, we also study functionals of the form

$$\int_\Omega f(Ddu)$$

and the corresponding variational problems for a given function f. If f is convex and satisfies suitable coercivity conditions, then the minimization of such a functional is not much different from E_p. However, similarly to other problems in the calculus of variations, the convexity can be relaxed, even though in contrast to more classical problems, the curvature of the target manifold will become important if we do so. We consider the notion of polyconvexity and how it fits into the framework developed in this paper. Under certain technical assumptions, we show that the direct method can still be used for polyconvex functions.

2 Maps with p-integrable intrinsic Hessian

We first consider some problems that arise only in the case $p < 2$. For this reason, we assume that $p \in (1,2]$ for the moment. Let $u \in W^{1,p}(\Omega, N)$. We denote by $p' = \frac{p}{p-1}$ the conjugate exponent to p and we set $p^* = \frac{mp}{m-p}$. For $y \in N$ we denote by $P(y)$ the orthonormal projection from $\mathbb{R}^n$ onto the tangent space T_yN of N at y. We define the Sobolev spaces

$$W^{k,p}(\Omega, N) = \{u \in W^{k,p}(\Omega, \mathbb{R}^n) : u(x) \in N \text{ almost everywhere.}\}.$$

We introduce the shorthand notation

$$\Gamma = u^{-1}TN$$

to denote the pulled back tangent bundle whose fibers are given by $\Gamma_x = T_{u(x)}N$ for all $x \in \Omega$. We write $X \in L^s(\Omega, \Gamma)$ if $X \in L^s(\Omega, \mathbb{R}^n)$ with $X(x) \in \Gamma_x$ for almost every $x \in \Omega$. For $Z \in L^{p'}(\Omega, \Gamma)$ and $\alpha = 1, ..., m$, we define the distributions

$$D_\alpha Z := \partial_\alpha Z + A(u)(\partial_\alpha u, Z). \tag{2}$$

Since $u \in W^{1,p}$, by Hölder's inequality this is an element of $(W^{1,p})' + L^1$. For sufficiently regular Z and u, the operation D_α defined via (2) is just the covariant derivative in the pulled back tangent bundle $u^{-1}TN$. We want a similar concept for sections of Γ that belong only to $L^p(\Omega, \Gamma)$, and then formula (2) has no direct interpretation. The idea is to consider derivatives of $(1+|Z|^2)^{\frac{p-2}{2}} Z$ instead and define $D_\alpha Z$ in terms of these. But we need some technical observations first.

Lemma 2.1 *Let $u \in W^{1,p}(\Omega, N)$ and $Z \in L^{p'}(\Omega, \Gamma)$. Then the distribution (2) satisfies*

$$(D_\alpha Z)(\varphi) = -\int_\Omega Z \cdot \partial_\alpha \left(P(u)\varphi\right) \text{ for all } \varphi \in C_0^\infty(\Omega, \mathbb{R}^n). \tag{3}$$

Proof. Let $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$. As $u \in W^{1,p}$ and P is smooth and bounded, we can apply the chain rule for Sobolev functions to find

$$\partial_\alpha \left(P(u)\right) = (\partial_\alpha u_i)(\partial_i P)(u)\varphi.$$

By [9, Theorem 1 (v) in Section 2.12.3] we have

$$(\partial_i P_{jk})(y) v_i w_k = -A_j(y)(v, w) \text{ for all } y \in N,\ v, w \in T_yN.$$

Since Z is a section of Γ we see that

$$Z \cdot \partial_\alpha \left(P(u)\right) \varphi = -A(u)(\partial_\alpha u, Z) \cdot \varphi.$$

Hence, using $P(u)Z = Z$ and the Leibniz rule, we conclude that

$$-Z \cdot \partial_\alpha (P(u)\varphi) = -Z \cdot \partial_\alpha \varphi + A(u)(\partial_\alpha u, Z) \cdot \varphi.$$

The integral over the right-hand side agrees with the action of the distribution (2) on the test function φ. □

Define $\eta_p : [0, \infty) \to [0, \infty)$ by

$$\eta_p(t) = (1 + t^2)^{\frac{p-2}{2}} \tag{4}$$

and define $M_p : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ by

$$M_p(y) = |y|\eta_p'(|y|) \left(\frac{y}{|y|} \otimes \frac{y}{|y|} \right) + \eta_p(|y|)I, \tag{5}$$

where I denotes the $n \times n$ identity matrix.

Lemma 2.2 *Let $X \in L^p(\Omega, \Gamma)$. Then $\eta_p(|X|)X \in L^{p'}(\Omega, \Gamma)$ and $|X|\eta_p'(|X|) \in L^\infty(\Omega)$. Moreover, $M_p(X) \in L^\infty(\Omega, \mathbb{R}^{n \times n})$ with $\det M_p(X) > 0$ almost everywhere on Ω.*

Proof. We have

$$\eta_p'(t) = \frac{(p-2)t}{1+t^2}\, \eta_p(t). \tag{6}$$

Hence

$$|t\eta_p'(t)| \le (2-p)\eta_p(t) \text{ for all } t \in [0, \infty). \tag{7}$$

If $X \in L^p(\Omega, \Gamma)$, then

$$\eta_p(|X|)|X| \le C|X|^{p-1} \in L^{p'}(\Omega). \tag{8}$$

The assertions about $\eta_p(|X|)X$ and $|X|\eta_p'(|X|)$ follow from (8), (7) and because $\eta_p \in L^\infty$. Hence $M_p(X) \in L^\infty$. Finally, for all $y \in \mathbb{R}^n$ the matrix $M_p(y)$ is invertible (and orientation preserving) by (7) and because $|p-2| < 1$. □

The following result is Theorem 2.1.4 in [10].

Lemma 2.3 *Let $p \ge 1$ and let $f \in L^p(\Omega)$. Then $f \in W^{1,p}(\Omega)$ if and only if the following is true: The function f (has a representative that) is absolutely continuous on almost every line segment in Ω which is parallel to some coordinate axis, and all partial derivatives of f (which consequently exist almost everywhere) belong to $L^p(\Omega)$.*

By Lemma 2.2, the expression

$$D(\eta_p(|X|)X) = \nabla\Big((\eta_p(|X|)X\Big) + A(u)\Big(\nabla u, \eta_p(|X|)X\Big),$$

obtained by taking $Z = \eta_p(|X|)X$ in (2) is well-defined as a distribution. On the other hand, a formal calculation using $DX = P(u)\nabla X$ gives:

$$D(\eta_p(|X|)X) = M_p(X)DX, \tag{9}$$

where M_p is as in (5). The formal derivation of (9) is justified if X is absolutely continuous along almost every coordinate line. Hence we have:

Lemma 2.4 *If a section $X \in L^1(\Omega, \Gamma)$ is absolutely continuous on almost every line segment in Ω which is parallel to some coordinate axis, then for all $\alpha = 1, ..., m$,*

$$\partial_\alpha \left(\eta_p(|X|)X\right) = M_p(X)(\partial_\alpha X) \text{ almost everywhere on } \Omega. \tag{10}$$

In particular, by the definition of M_p,

$$P(u)\partial_\alpha \left(\eta_p(|X|)X\right) = M_p(X)(P(u)\partial_\alpha X) \text{ almost everywhere on } \Omega. \tag{11}$$

Together with Lemma 2.6, the formula (11) motivates the following definition:

Definition 2.5 *A section $X \in L^p(\Omega, \Gamma)$ is said to belong to $W^{1,p}(\Omega, \Gamma)$ if there exist sections $Y_1, ..., Y_m \in L^p(\Omega, \Gamma)$ such that, for all $\alpha = 1, ..., m$,*

$$M_p(X)Y_\alpha = D_\alpha \left(\eta_p(|X|)X\right) \text{ on } \Omega \tag{12}$$

in the sense of distributions. If this is the case, then we write $D^\Gamma_\alpha X := Y_\alpha$ for $\alpha = 1, ..., m$.

Remarks.

(i) The left-hand side of (12) is well-defined in L^p because $M_p(X) \in L^\infty$, and the right-hand side of (12) is well-defined as a distribution because $\eta_p(|X|)X \in L^{p'}$.

(ii) If $X \in W^{1,p}(\Omega, \Gamma)$ then $D^\Gamma \left(\eta_p(|X|)X\right) \in L^p$ because $M_p \in L^\infty$. However, the converse may fail because $\det M_p(X)$ may not be uniformly bounded away from zero.

We will usually write D instead of D^Γ when there is no danger of confusion. The following lemma is our main technical result about sections in $W^{1,p}(\Omega, \Gamma)$. A consequence of the following lemma is that DX does not depend on p.

Lemma 2.6 *Let $p \in (1, 2]$, let $u \in W^{1,p}(\Omega, N)$ and let $X \in W^{1,p}(\Omega, \Gamma)$. Then $|X| \in W^{1,p}(\Omega)$ with*

$$\partial_\alpha |X| = \chi_{\{X \neq 0\}} \frac{X}{|X|} \cdot D^\Gamma_\alpha X \text{ almost everywhere on } \Omega \tag{13}$$

for all $\alpha = 1, ..., m$. Moreover, X is absolutely continuous along almost every line segment in Ω which is parallel to some coordinate axis (hence the partial derivatives $\partial_\alpha X$ exist almost everywhere), and we have

$$D^\Gamma_\alpha X = P(u)\partial_\alpha X \text{ almost everywhere on } \Omega. \tag{14}$$

Proof. Since $M_p \in L^\infty$, we see that $D_\alpha(\eta_p(|X|)X) \in L^1$ (even in L^p). Since $\eta_p(|X|)X \in L^{p'} \subset L^2$, by (2) we have

$$D_\alpha \left(\eta_p(|X|)X\right) = A(u) \left(\partial_\alpha u, \eta_p(|X|)X\right) + \partial_\alpha \left(\eta_p(|X|)X\right)$$

as distributions. We conclude that

$$\eta_p(|X|)X \in W^{1,1}, \tag{15}$$

because $A(u)(\partial_\alpha u, \eta_p(|X|)X) \in L^1$ by Hölder's inequality. By (15) we have

$$\partial_\alpha \left(\eta_p(|X|)|X|\right) = \frac{X}{|X|} \cdot D_\alpha \left(\eta_p(|X|)X\right) \tag{16}$$

almost everywhere on $\{X \neq 0\}$. As

$$(\eta_p(t)t)' = \frac{1+(p-1)t^2}{1+t^2}\eta_p(t)$$

is strictly positive for all $t \in \mathbb{R}$, the inverse H_p of the function $t \mapsto \eta_p(t)t$ exists. We have

$$|X| = H_p(\eta_p(|X|)|X|) \text{ almost everywhere.}$$

Moreover, $H_p \in C^\infty(\mathbb{R})$ and

$$H_p'\Big(\eta_p(|X|)|X|\Big)\nabla\Big(\eta_p(|X|)|X|\Big) \in L^p(\Omega). \tag{17}$$

In fact, we have

$$H_p'(t\eta_p(t)) = \frac{1+t^2}{1+(p-1)t^2}\frac{1}{\eta_p(t)}.$$

And, setting $Y_\alpha = D_\alpha X$, by (16) and (12)

$$\left|\nabla\Big(\eta_p(|X|)|X|\Big)\right| \leq C|M_p(X)Y| \leq C\eta_p(|X|)|Y|$$

pointwise almost everywhere. Hence the left-hand side of (17) is dominated by a constant times $|Y|$.

By (17) we can apply, e.g., Theorem 3.1.9 in [5] to conclude that

$$|X| = H_p\left(\eta_p(|X|)|X|\right) \in W^{1,1}(\Omega)$$

with

$$\nabla|X| = H_p'\Big(\eta_p(|X|)|X|\Big)\nabla\Big(\eta_p(|X|)|X|\Big) \text{ almost everywhere.} \tag{18}$$

In particular, $|X| \in W^{1,p}$ by (17). Combining (18) with (16), (12) and with the definition of M_p, we deduce (13).

Since $|X| \in W^{1,1}$, also $\left(\frac{1}{\eta_p}\right)(|X|) \in W^{1,1}$ because $\frac{1}{\eta_p} \in C^1(\mathbb{R})$ with $\left(\frac{1}{\eta_p}\right)' \in L^\infty(\mathbb{R})$. Therefore

$$X = \left(\frac{1}{\eta_p}\right)(|X|) \cdot \eta_p(|X|)X$$

belongs to $W^{1,1}\cdot W^{1,1}$. Hence X is absolutely continuous along almost every line segment in Ω which is parallel to some coordinate axis. Since $\eta_p \in C^1(\mathbb{R})$ with $\eta_p' \in L^\infty(\mathbb{R})$, also $\eta_p(|X|)$ is absolutely continuous along almost every segment in Ω that is parallel to some coordinate axis. Hence we can apply the product rule to $\eta_p(|X|)X$ to find

$$\partial_\alpha\Big(\eta_p(|X|)X\Big) = \eta_p'(|X|)(\partial_\alpha|X|)X + \eta_p(|X|)\partial_\alpha X$$

almost everywhere. Hence

$$P(u)\partial_\alpha\Big(\eta_p(|X|)X\Big) = M_p(X)P(u)\partial_\alpha X$$

almost everywhere. But since $\eta_p(|X|)X \in L^{p'}$, we know that the left-hand side of this equation agrees with $D_\alpha\Big(\eta_p(|X|)X\Big)$. Hence (14) follows from (12) and the fact that $M_p(y)$ is invertible for all $y \in \mathbb{R}^n$, cf. Lemma 2.2. $\Box$

Corollary 2.7 *Let $u \in W^{1,p}(\Omega, N)$ and $X \in W^{1,p}(\Omega, \Gamma)$. Then, for all $\alpha = 1, ..., m$,*

$$|\partial_\alpha |X|| \leq |\partial_\alpha X| \text{ almost everywhere on } \Omega. \tag{19}$$

In particular,

$$\|X\|_{L^{p^*}(\Omega)} \leq C\left(\|X\|_{L^p(\Omega)} + \|DX\|_{L^p(\Omega)}\right) \tag{20}$$

and

$$\|X\|_{L^p(\Omega)} \leq C\left(\|X\|_{L^p(\partial\Omega)} + \|DX\|_{L^p(\Omega)}\right). \tag{21}$$

Proof. Equation (19) follows from (13). The remaining claims are then obtained by applying Sobolev's and Poincaré's inequalities to $|X|$.

$\Box$

We can prove a Leibniz rule for the intrinsic derivative:

Corollary 2.8 *Let $p \in (1, 2]$, let $u \in W^{1,p}(\Omega, N)$ and let $X \in W^{1,p}(\Omega, \Gamma)$ and $Y \in W^{1,p'}(\Omega, \Gamma)$. Then $X \cdot Y \in W^{1,1}(\Omega)$ and, for all $\alpha = 1, ..., m$,*

$$\partial_\alpha(X \cdot Y) = (D_\alpha X) \cdot Y + X \cdot (D_\alpha Y) \text{ almost everywhere.} \tag{22}$$

Remark. In the terminology of differential geometry, equation (22) means that D^Γ is a metric connection. Although this is obvious from the definition if u and X are smooth, the statement needs to be verified for the weak version of the covariant derivative.

Proof. By Lemma 2.6 we know that both X and Y are absolutely continuous along almost every coordinate line. Hence so is $X \cdot Y$. Thus we can apply the usual Leibniz rule along almost every coordinate line to find $\partial_\alpha(X \cdot Y) = (\partial_\alpha X) \cdot Y + (\partial_\alpha Y) \cdot X$. By Lemma 2.6 the right-hand side agrees almost everywhere with

$$(D_\alpha X) \cdot Y + X \cdot (D_\alpha Y).$$

And this expression belongs to L^1. Hence the claim follows from Lemma 2.3. $\Box$

The following lemma is the intrinsic counterpart of the 'if'-part of Lemma 2.3.

Lemma 2.9 *Let $p \in (1, 2]$, let $u \in W^{1,p}(\Omega, N)$, let $X \in L^p(\Omega, \Gamma)$ and suppose that X is absolutely continuous on almost every line segment in Ω which is parallel to some coordinate axis and such that*

$$P(u)\,(\partial_\alpha X) \in L^p(\Omega, \mathbb{R}^n) \text{ for all } \alpha = 1, ..., m.$$

Then $X \in W^{1,p}(\Omega, \Gamma)$.

Proof. If a section $X \in L^1(\Omega, \Gamma)$ is absolutely continuous on almost every line segment parallel to some coordinate axis, then (11) is satisfied. On the other hand, for any section $Z \in L^{p'}(\Omega, \Gamma)$, the distribution $D_\alpha Z$ given via (2) satisfies (3).
In order to prove the lemma, set $Z = \eta_p(|X|)X$. Then Z is absolutely continuous along almost every segment in Ω parallel to some coordinate axis. Moreover, the right-hand side of (11) is in L^p because $P(u)(\partial_\alpha X) \in L^p$ by hypothesis and because M_p is bounded, cf. Lemma 2.2. Hence $P(u)(\partial_\alpha Z) \in L^p$. Thus we can use Fubini's Theorem and integrate by parts on segments in (3) to conclude that

$$D_\alpha Z = P(u)(\partial_\alpha Z) \text{ as distributions,}$$

where the right-hand side is computed almost everywhere. Since $P(u)(\partial_\alpha Z) = M_p(X)\,(P(u)\partial_\alpha X)$ by (11), we conclude that indeed $X \in W^{1,p}(\Omega, \Gamma)$, with $D_\alpha X = P(u)(\partial_\alpha X)$. □

3 Existence of minimizers by the direct method

If $u \in W^{1,p}(\Omega, N)$ then its partial derivatives $\partial_\alpha u$ belong to $L^p(\Omega, \Gamma)$. Hence we can apply Definition 2.5 to define $D\partial_\alpha u$.

Definition 3.1 *We denote by $W^{2,p}_N(\Omega)$ the set of all $u \in W^{1,p}(\Omega, N)$ such that $\partial_\alpha u \in W^{1,p}(\Omega, \Gamma)$ for all $\alpha = 1, ..., m$. On this space we define*

$$\|D^\Gamma du\|^p_{L^p(\Omega)} := \sum_{\alpha=1}^{m} \|D^\Gamma \partial_\alpha u\|^p_{L^p(\Omega)},$$

and we set $E_p(u) = \|D^\Gamma du\|^p_{L^p(\Omega)}$.

The purpose of this section is to prove the existence of minimizers for E_p within the class $W^{2,p}_N(\Omega)$. We will need the following lemma.

Lemma 3.2 *Let $p \in (1, 2)$, let $u, u_k \in W^{1,p}(\Omega, N)$, set $\Gamma_k = u_k^{-1}TN$ and denote by D^{Γ_k} the covariant derivative on this bundle. Let $X \in W^{1,p}(\Omega, \Gamma)$ and $X_k \in W^{1,p}(\Omega, \Gamma_k)$, and assume that*

$$u_k \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega, \mathbb{R}^n)$$

and

$$X_k \rightharpoonup X \text{ weakly in } L^p(\Omega, \mathbb{R}^n), \tag{23}$$

and that

$$\limsup_{n\to\infty} \|D^{\Gamma_k} X_k\|_{L^p(\Omega)} < \infty. \tag{24}$$

Then there is a subsequence such that

$$X_k \to X \text{ strongly in } L^q(\Omega) \text{ for all } q < p^*. \tag{25}$$

If, in addition,

$$u_k \to u \text{ strongly in } W^{1,p}(\Omega, \mathbb{R}^n),$$

then we have

$$D_\alpha^{\Gamma_k} X_k \rightharpoonup D_\alpha^{\Gamma} X \text{ weakly in } L^p(\Omega, \mathbb{R}^n) \text{ for all } \alpha = 1, ..., m. \tag{26}$$

Proof. Since the sequence (X_k) is uniformly bounded in L^p, the sequence $Z_k = \eta_p(|X_k|)X_k$ is uniformly bounded in $L^{p'}$. Hence

$$\nabla Z_k = M_p(X_k) D^{\Gamma_k} X_k - A(u)(\nabla u_k, Z_k)$$

are uniformly bounded in L^1 by the hypotheses on u_k and on X_k, and because $M_p \in L^\infty$. It follows that there exists $Z \in L^{p'}$ such that, for a subsequence, we have in particular

$$Z_k \to Z \text{ weakly in } L^{p'} \text{ and pointwise almost everywhere.} \tag{27}$$

In particular, denoting by $F_p : \mathbb{R}^n \to \mathbb{R}^n$ the (continuous) inverse of the map $y \mapsto \eta_p(|y|)y$, we have

$$X_k = F_p(Z_k) \to F_p(Z) \text{ pointwise almost everywhere,} \tag{28}$$

hence in measure, hence $X_k \rightharpoonup F_p(Z)$ in L^p because the sequence (X_k) is bounded in L^p. Thus (23) implies that $X = F_p(Z)$, so $X_k \to X$ pointwise almost everywhere by (28).
Thus also

$$|X_k| \to |X| \text{ pointwise almost everywhere.} \tag{29}$$

But the hypotheses on X_k and on $D^{\Gamma_k} X_k$ together with (19) imply that both X_k and $\nabla |X_k|$ are uniformly bounded in L^p. Applying Rellich's Theorem to $|X_k|$ therefore implies that, after passing to a subsequence, $|X_k|$ converges strongly in L^p. By (29) the limit must be $|X|$. In particular,

$$\|X_k\|_{L^p} \to \|X\|_{L^p}.$$

Together with the weak convergence this implies that $X_k \to X$ strongly in L^p. To conclude the proof of (25), we apply (20) and interpolate.
Now suppose that $u_k \to u$ strongly in $W^{1,p}$. Then we deduce directly from the definition (2) and from (27) that

$$D^{\Gamma_k} Z_k \overset{*}{\rightharpoonup} DZ \text{ as distributions.}$$

On the other hand,

$$D^{\Gamma_k} Z_k = M_p(X_k) D^{\Gamma_k} X_k$$

by (12). And $M_p(X_k) \to M_p(X)$ strongly in $L^{p'}$ by (25), while by (24) there is a subsequence such that $D^{\Gamma_k} X_k$ converges weakly in L^p to some map $Y \in L^p(\Omega, \mathbb{R}^n)$. It follows that $D^{\Gamma_k} Z_k \overset{*}{\rightharpoonup} M_p(X)Y$ as distributions. Hence $M_p(X)Y = DZ$, that is, $Y = DX$. □

When $p \geq 2$ then the definition of DX shows that $DX \in L^p$ implies $X \in W^{1,1}$, so it is clear that sections $X \in W^{1,2}(\Omega, \Gamma)$ admit traces on $\partial\Omega$, cf. [6]. If $X \in W^{1,p}(\Omega, \Gamma)$ for some $p \in (1, 2)$, then one can still define the trace of X as follows: We have $\eta_p(|X|)X \in W^{1,1}$ (cf. (15)), so this section admits traces on $\partial\Omega$. Now we define the trace of X in the obvious way, namely by applying pointwise the inverse of the map

$$\mathbb{R}^n \to \mathbb{R}^n, \ y \mapsto \eta_p(|y|)y$$

to the trace of $\eta_p(|X|)X$. The assertion of the next theorem is to be understood in that sense.

Theorem 3.3 *Let $p \in (1,2)$ and let $u_0 \in W^{2,p}_N(\Omega)$ be given. Then the functional E_p attains its minimum among all $u \in W^{2,p}_N(\Omega)$ satisfying $(u, du) = (u_0, du_0)$ on $\partial\Omega$ in the sense of traces.*

Proof. Let $u_k \in W^{2,p}_N(\Omega)$ be such that

$$(u_k, du_k) = (u_0, du_0) \text{ on } \partial\Omega \tag{30}$$

in the trace sense as defined above, and assume that (u_k) is a minimizing sequence for E_p within this class. Then $E_p(u_k)$ is uniformly bounded, i.e.,

$$\limsup_{n\to\infty} \|D^{\Gamma_k} du_k\|_{L^p(\Omega)} < \infty. \tag{31}$$

Now (21) together with the boundary conditions implies that $|du_k|$ is uniformly bounded in L^p. Since u_k is uniformly bounded in L^∞ by compactness of N, we conclude that u_k is uniformly bounded in $W^{1,p}$. Hence a subsequence converges weakly in this space. Lemma 3.2 then implies that

$$u_k \to u \text{ strongly in } W^{1,p}. \tag{32}$$

Thus we can apply the last part of Lemma 3.2 to conclude that

$$D^{\Gamma_k} du_k \rightharpoonup Ddu \text{ weakly in } L^p.$$

By weak lower semicontinuity of the L^p-norm we conclude that $E_p(u)$ does not exceed the limit inferior of $E_p(u_k)$.
To conclude that u is the sought-for minimizer, it remains to show that u satisfies the boundary conditions. Define $X_k = \eta_p(|du_k|)du_k$. Then X_k and $D^{\Gamma_k} X_k$ are uniformly bounded in $L^{p'}$. Hence (after passing to subsequences) Lemma 3.2 implies

$$X_k \to X \text{ strongly in } L^{p'} \tag{33}$$

$$D^{\Gamma_k} X_k \rightharpoonup DX \text{ weakly in } L^{p'}. \tag{34}$$

Since $p' \geq 2$, the formula (2) is satisfied by X_k, i.e.,

$$\nabla X_k = D^{\Gamma_k} X_k - A(u_k)(\nabla u_k, X_k). \tag{35}$$

But by (20), the formulae (33), (34) and (31), (32) ensure that $|\nabla u_k|$ is uniformly bounded in L^{p^*} and that X_k is uniformly bounded in $L^{(p')^*}$. Hence the second term on the right-hand side of (35) is uniformly bounded in L^q for some $q > 1$. Thus (35) implies a uniform $W^{1,q}$-bound on X_k. Hence a subsequence converges weakly in $W^{1,q}$, and the traces of the X_k on $\partial\Omega$ converge to the trace of X on $\partial\Omega$. Since $X_k = \eta_p(|du_0|)du_0$ $\mathcal{H}^{m-1}$ almost everywhere on $\partial\Omega$, this implies that

$$X = \eta_p(|du_0|)du_0 \text{ on } \partial\Omega \tag{36}$$

in the trace sense. But (after passing to subsequences) (33) implies that $X_k \to X$ pointwise almost everywhere and (32) implies that $du_k \to du$ pointwise almost everywhere. Thus $X = \eta_p(|du|)du$, and so (36) means that $du = du_0$ on $\partial\Omega$ in the trace sense. Finally, from (32) it is clear that $u = u_0$ on $\partial\Omega$. $\square$

Remark. For $p \geq 2$, the Sobolev space $W^{2,p}_N(\Omega)$ can be defined more directly using formula (2). With the same arguments as in [6], we see that the statement of Theorem 3.3 carries over to this case.

4 Functionals with polyconvex energy densities

With observations similar to the previous sections, we can also study functionals such as

$$F(u) = \int_\Omega f(Ddu)\,dx$$

for more general energy densities $f : \mathbb{R}^m \otimes \mathbb{R}^m \otimes TN \to \mathbb{R}$. Since our target manifold is embedded in $\mathbb{R}^n$, we can always extend f to the ambient space. For convenience, we assume that we have no direct dependence on the values of u, i.e, we can represent the functional in terms of a function $f : \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}$ in the following. We also assume that f is continuous.

The previous arguments still work exactly the same way if we have coercivity and weak lower semicontinuity of the functional $\tilde{F} : L^p(\Omega, \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^n) \to \mathbb{R}$ given by

$$\tilde{F}(X) = \int_\Omega f(X)\,dx.$$

It is well-known that the latter is equivalent to convexity of f under reasonable conditions [2, Theorem 5.14]. In particular, the following holds true.

Theorem 4.1 *Let $p \in (1, \infty)$. Suppose that $f : \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}$ is convex and there are two constants $c, C > 0$ such that*

$$c|X|^p - C \leq f(X) \leq C(|X|^p + 1)$$

for all $X \in \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^n$. Let $u_0 \in W^{2,p}_N(\Omega)$. Then F attains its minimum among all $u \in W^{2,p}_N(\Omega)$ satisfying $(u, du) = (u_0, du_0)$ on $\partial\Omega$.

It is also well-known that lower semicontinuity in Sobolev spaces is related to the weaker notion of quasiconvexity rather than convexity, thus by analogy, we expect that the hypotheses in this theorem can be relaxed, cf. also [1]. We do not have any results for functions that are merely quasiconvex, but we can work with the intermediate concept of polyconvexity. In the simplest case of a function $f : \mathbb{R}^m \otimes \mathbb{R}^m \to \mathbb{R}$, this is tantamount to convexity of $f(X)$ in the minors of X (including the $(m \times m)$- and (1×1)-minors, i.e., the determinant and the entries of X). In the more general situation considered here, it is easier to express the corresponding property in terms of differential forms, and for this reason we now introduce some notation.

Suppose that $u \in W^{2,p}_N(\Omega)$ and $\Gamma = u^{-1}TN$. Let $I \in \mathbb{N}$ and let

$$X = X_{\alpha_1 \dots \alpha_I} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_I}$$

be an I-form with values in Γ; i.e., the coefficients $X_{\alpha_1 \dots \alpha_I}$ are sections of Γ. Then we define

$$d^\Gamma X = D^\Gamma_\beta X_{\alpha_1 \dots \alpha_I} dx^\beta \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_I}. \tag{37}$$

We use the symbol d^Γ here rather than D^Γ (in contrast to the common practice of reusing the symbol for the connection on Γ), because the expression $D^\Gamma du$ would become ambiguous otherwise. With the usual exterior derivative d given by

$$dX = \partial_\beta X_{\alpha_1 \dots \alpha_I} \; dx^\beta \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_I}$$

and with the $(I+1)$-form

$$A(u)(du, X) = A(u)(\partial_\beta u, X_{\alpha_1 \dots \alpha_I}) \; dx^\beta \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_I}, \tag{38}$$

the definition (37) of d^Γ can be concisely written as

$$d^\Gamma X = dX + A(u)(du, X), \tag{39}$$

provided that the last term is well-defined.

The connection d^Γ has curvature given in terms of the Riemann curvature tensor R on N. More precisely, we have the following.

Lemma 4.2 *Suppose that $p \geq \max\{2, \frac{3m}{m+3}\}$ and $u \in W^{2,p}_N(\Omega)$. Let*

$$X = X_{\alpha_1 \dots \alpha_I} dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_I}$$

with $X_{\alpha_1 \dots \alpha_I} \in W^{1,p}(\Omega, \Gamma)$. Then

$$(d^\Gamma)^2 X = \frac{1}{2} R(u)(\partial_\beta u, \partial_\gamma u) X_{\alpha_1 \dots \alpha_I} dx^\beta \wedge dx^\gamma \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_I} \tag{40}$$

in the sense of distributions.

This statement is to be understood as follows. Since $p \geq 2$, we have

$$D^\Gamma_\beta X_{\alpha_1 \dots \alpha_I} = \partial_\beta X_{\alpha_1 \dots \alpha_I} + A(u)(\partial_\beta u, X_{\alpha_1 \dots \alpha_I}).$$

Hence the distribution $(d^\Gamma)^2 X$ is given by

$$\begin{aligned} (d^\Gamma)^2 X = \Big\{ &\partial_\gamma \partial_\beta X_{\alpha_1 \dots \alpha_I} + \partial_\gamma \left(A(u)(\partial_\beta u, X_{\alpha_1 \dots \alpha_I}) \right) \\ &+ A(u)(\partial_\gamma u, D^\Gamma_\beta X_{\alpha_1 \dots \alpha_I}) \Big\} \, dx^\gamma \wedge dx^\beta \wedge dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_I}. \end{aligned} \tag{41}$$

Since $|\nabla u|$, $|X|$, $|D^\Gamma X| \in L^p(\Omega)$ and $p \geq 2$, each term on the right-hand side of (41) is indeed well-defined as a distribution. Moreover, by the Sobolev embedding theorem (cf. (20)), we have $|X|, |\nabla u| \in L^{mp/(m-p)}(\Omega)$. As $\frac{3(m-p)}{mp} \leq 1$, it follows from Hölder's inequality that the right-hand side of (40) is integrable. Hence equation (40) is indeed meaningful in the sense of distributions.

Proof. It suffices to show that for all $X \in W^{1,p}(\Omega, \Gamma)$,

$$D^\Gamma_\alpha D^\Gamma_\beta X - D^\Gamma_\beta D^\Gamma_\alpha X = R(u)(\partial_\beta u, \partial_\alpha u)X. \tag{42}$$

This follows by a direct calculation if everything is smooth. But since we cannot differentiate twice here, we need some more work.

The underlying idea of the proof is to approximate u and X by a smooth map and a smooth section of the corresponding vector bundle, respectively. However, the space $C^\infty(\Omega, N)$ is not dense in $W^{2,p}_N(\Omega)$ in general. On the other hand, in order to verify (42), we need only consider the restrictions to two-dimensional planes parallel to the x^α- and x^β-axes. After applying both sides of the equation to a test function, the identity then follows from Fubini's theorem.

Assume for simplicity that $\alpha = 1$ and $\beta = 2$. Then for almost all points $x' = (x^3, \ldots, x^m) \in \mathbb{R}^{m-2}$, the map $u_{x'}(x^1, x^2) = u(x^1, \ldots, x^m)$ belongs to $W^{2,p}_N(\Omega_{x'})$, where

$$\Omega_{x'} = \{(x^1, x^2) \in \mathbb{R}^2 \colon (x^1, \ldots, x^m) \in \Omega\}.$$

In fact, by the usual fine properties of Sobolev maps (e.g. by Lemma 2.3), for almost every $x' \in \mathbb{R}^{m-2}$ we have that $u_{x'} \in W^{1,p}(\Omega_{x'}, \mathbb{R}^n)$. By Lemma 2.6 we have that ∇u is absolutely continuous along almost every coordinate line in Ω (hence for almost every x' it is absolutely continuous along almost every coordinate line in $\Omega_{x'}$) and $|P(u)\nabla^2 u| \in L^p(\Omega)$. In particular, by Fubini's Theorem,

$$|P(u)\left(\nabla^2 u\right)| \in L^p(\Omega_{x'})$$

for almost every $x' \in \mathbb{R}^{m-2}$. Thus Lemma 2.9 shows that indeed $u_{x'} \in W^{2,p}_N(\Omega_{x'})$ for almost every $x' \in \mathbb{R}^{m-2}$.
Similarly, $X_{x'}(x^1, x^2) = X(x^1, \ldots, x^m)$ belongs to $W^{1,p}(\Omega, \Gamma_{x'})$, where $\Gamma_{x'} = u_{x'}^{-1}TN$. Since $p \geq 2$ and since $\Omega_{x'}$ is two-dimensional, it follows from (20) that $X_{x'} \in L^q(\Omega_{x'}, \mathbb{R}^n)$ and $u_{x'} \in W^{1,q}(\Omega_{x'}, \mathbb{R}^n)$ for any $q > 2$. In particular, (2) can be applied, so

$$\partial_\alpha X_{x'} = D^{\Gamma_{x'}}_\alpha X_{x'} - A(u_{x'})(\partial_\alpha u_{x'}, X_{x'}).$$

Choosing q large enough, the second term on the right-hand side belongs to L^p, hence

$$X_{x'} \in W^{1,p}(\Omega_{x'}, \mathbb{R}^n). \tag{43}$$

Since $u_{x'} \in W^{1,q}$ with $q > 2$, we see that $u_{x'}$ is continuous, so there exist $\widetilde{u}^{(k)} \in C^\infty(\Omega_{x'}, \mathbb{R}^n)$ converging uniformly on $\Omega_{x'}$ and strongly in $W^{1,q}(\Omega_{x'}, \mathbb{R}^n)$ to $u_{x'}$ as $k \to \infty$. Denote by π_N the nearest point projection from a tubular neighbourhood of N onto N. Then the maps

$$u^{(k)} = \pi_N(\widetilde{u}^{(k)})$$

belong to $C^\infty(\Omega_{x'}, N)$, and they converge to $u_{x'}$ strongly in $W^{1,q}(\Omega_{x'}, \mathbb{R}^n)$ and uniformly on $\Omega_{x'}$.

By (43) there exist $\widetilde{X}^{(k)} \in C^\infty(\Omega_{x'}, \mathbb{R}^n)$ be such that $\widetilde{X}^{(k)} \to X$ strongly in $W^{1,p}(\Omega_{x'}, \mathbb{R}^n)$. Set

$$X^{(k)} = P(u^{(k)})\widetilde{X}^{(k)}.$$

As $u^{(k)}$ and $\widetilde{X}^{(k)}$ are smooth, we have $X^{(k)} \in C^\infty(\Omega_{x'}, \Gamma^{(k)})$, where $\Gamma^{(k)} = (u^{(k)})^{-1}TN$. Thus

$$D_1^{\Gamma^{(k)}} D_2^{\Gamma^{(k)}} X^{(k)} - D_2^{\Gamma^{(k)}} D_1^{\Gamma^{(k)}} X^{(k)} = R(u^{(k)})(\partial_1 u^{(k)}, \partial_2 u^{(k)}) X^{(k)} \tag{44}$$

for all k. Since, e.g.,

$$D_1^{\Gamma^{(k)}} D_2^{\Gamma^{(k)}} X^{(k)} = \partial_1\partial_2 X^{(k)} + \partial_1 \Big(A(u^{(k)})(\partial_2 u^{(k)}, X^{(k)}) \Big) + A(\partial_1 u^{(k)}, D_2^{\Gamma^{(k)}} X^{(k)}),$$

and since $X^{(k)} \to X$ in $W^{1,p}(\Omega_{x'}, \mathbb{R}^n)$, the left-hand side of (44) converges in distributions to the left-hand side of (42). Choosing q large enough, the right-hand of (44) side converges strongly in L^1 to the right-hand side of (42). Hence (42) is indeed satisfied. □

Now consider a map $u \in W^{2,p}_N(\Omega)$ and set $\Gamma = u^{-1}TN$ again. Let $X_1, \ldots, X_I \in W^{1,p}(\Omega, \Gamma)$ and consider the I-form

$$d^\Gamma X_1 \wedge \ldots \wedge d^\Gamma X_I = D^\Gamma_{\alpha_1} X_1 \otimes \cdots \otimes D^\Gamma_{\alpha_I} X_I \, dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_I}$$

with values in $\Gamma \otimes \cdots \otimes \Gamma$ (with I factors). Using Lemma 4.2 and writing

$$R(u)(du, du)X_i = R(u)(\partial_\alpha u, \partial_\beta u)X_i \; dx^\alpha \wedge dx^\beta,$$

we compute

$$\begin{aligned} d^\Gamma X_1 \wedge \ldots \wedge d^\Gamma X_I = {} & d^\Gamma (X_1 \otimes d^\Gamma X_2 \wedge \ldots \wedge d^\Gamma X_I) \\ & - \frac{1}{2} X_1 \otimes R(u)(du, du) X_2 \wedge d^\Gamma X_3 \wedge \ldots \wedge d^\Gamma X_I \\ & + \frac{1}{2} X_1 \otimes d^\Gamma X_2 \wedge R(u)(du, du) X_3 \wedge d^\Gamma X_4 \wedge \ldots \wedge d^\Gamma X_I \\ & - \cdots \pm \frac{1}{2} X_1 \otimes d^\Gamma X_1 \wedge \ldots \wedge d^\Gamma X_{I-1} \wedge R(u)(du, du) X_I, \end{aligned} \tag{45}$$

provided that p is sufficiently large. We use this formula for $X_i = \partial_{\gamma_i} u$ for a certain multi-index $\gamma = (\gamma_1, \ldots, \gamma_I)$. Then the forms

$$d^\Gamma \partial_{\gamma_1} u \wedge \ldots \wedge d^\Gamma \partial_{\gamma_I} u \tag{46}$$

are a natural generalization of the $(I \times I)$-minors of the Hessian of a function. Note that (45) generalizes the well-known fact that such minors can be written as a divergence.

In order to keep the following statement simple, we now consider a functional that depends, apart from $D^\Gamma du$, only on (46) for some fixed γ. It is not difficult to see that our method can be extended to include differential forms of different degrees, and lower order terms as well. Due to the curvature terms in (45), however, we have some restrictions on the values of p.

Theorem 4.3 *Let $I \in \{2, \ldots, m\}$ and $\gamma \in \{1, \ldots, m\}^I$. Suppose that $p > \max\{I, \frac{(I+2)m}{m+4}\}$. Let $f : \Lambda^I \mathbb{R}^m \otimes \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n \times \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^n \to \mathbb{R}$ be a convex function. Suppose there exist two constants $c, C > 0$ such that*

$$c|Y|^p - C \leq f(X,Y) \leq C(|X|^p + |Y|^p + 1)$$

for all $X \in \Lambda^I \mathbb{R}^m \otimes \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ and $Y \in \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^n$. Then the functional

$$F(u) = \int_\Omega f\left(d^\Gamma \partial_{\gamma_1} u \wedge \ldots \wedge d^\Gamma \partial_{\gamma_I} u, D^\Gamma du\right) \, dx$$

attains its minimum among all $u \in W_N^{2,p}(\Omega)$ satisfying $(u, du) = (u_0, du_0)$ on $\partial\Omega$.

Proof. Define $q = \frac{4p}{p+2-I}$. Suppose that $(u_k)_{k \in \mathbb{N}}$ is a minimizing sequence for the variational problem and set $\Gamma_k = u_k^{-1}TN$. It follows that the sequence is bounded in $W_N^{2,p}(\Omega)$, and we may assume without loss of generality that there exists a $u \in W_N^{2,p}(\Omega)$, satisfying the boundary conditions, such that $du_k \rightharpoonup du$ weakly in $L^p(\Omega, \mathbb{R}^m \otimes \mathbb{R}^n)$ and $D^{\Gamma_k} du_k \rightharpoonup D^\Gamma du$ weakly in $L^p(\Omega, \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^n)$. If $p < m$, then $u_k \to u$ strongly in $W^{1,p}$ by (32). Hence by (20) also $du_k \rightharpoonup du$ weakly in $L^{\frac{mp}{m-p}}$. If $p \geq m$ then $du_k \to du$ strongly in L^s for all $s > 1$. Since in the case $p < m$ we have $q < \frac{mp}{m-p}$, we conclude that in any case

$$du_k \to du \text{ strongly in } L^q(\Omega, \mathbb{R}^m \otimes \mathbb{R}^n). \tag{47}$$

We claim that

$$d^{\Gamma_k} \partial_{\gamma_1} u_k \wedge d^{\Gamma_k} \partial_{\gamma_2} u_k \rightharpoonup d^\Gamma \partial_{\gamma_1} u \wedge d^\Gamma \partial_{\gamma_2} u \tag{48}$$

weakly in $L^{p/2}(\Omega, \Lambda^2 \mathbb{R}^m \otimes \mathbb{R}^n \otimes \mathbb{R}^n)$. In fact, we have by (45)

$$\begin{aligned} d^{\Gamma_k} \partial_{\gamma_1} u_k \wedge d^{\Gamma_k} \partial_{\gamma_2} u_k =& d^{\Gamma_k}\left(\partial_{\gamma_1} u_k \otimes d^{\Gamma_k} \partial_{\gamma_2} u_k\right) \\ &- \frac{1}{2} \partial_{\gamma_1} u_k \otimes R(u_k)(du_k, du_k) \partial_{\gamma_2} u_k. \end{aligned} \tag{49}$$

Since

$$\frac{4}{q} = \frac{p+2-I}{p} \leq 1,$$

it is clear that

$$\partial_{\gamma_1} u_k \otimes R(u_k)(du_k, du_k) \partial_{\gamma_2} u_k \rightharpoonup \partial_{\gamma_1} u \otimes R(u)(du, du) \partial_{\gamma_2} u$$

weakly in $L^1(\Omega, \Lambda^2 \mathbb{R}^m \otimes \mathbb{R}^n \otimes \mathbb{R}^n)$. Using the formula

$$\begin{aligned} d^\Gamma(\partial_{\gamma_1} u \otimes d^\Gamma \partial_{\gamma_2} u) = d(\partial_{\gamma_1} u \otimes d^\Gamma \partial_{\gamma_2} u) + A(u)(du, \partial_{\gamma_1} u) \wedge d^\Gamma \partial_{\gamma_2} u& \\ + \partial_{\gamma_1} u \otimes A(u)(du, d^\Gamma \partial_{\gamma_2} u)&, \end{aligned}$$

we similarly see that

$$d^{\Gamma_k}(\partial_{\gamma_1} u_k \otimes d^{\Gamma_k} \partial_{\gamma_2} u_k) \rightharpoonup d^\Gamma(\partial_{\gamma_1} u \otimes d^\Gamma \partial_{\gamma_2} u)$$

weakly* in $(C_0^1(\Omega, \Lambda^2 \mathbb{R}^m \otimes \mathbb{R}^n \otimes \mathbb{R}^n))^*$. Since

$$d^{\Gamma_k} \partial_{\gamma_1} u_k \wedge d^{\Gamma_k} \partial_{\gamma_2} u_k$$

is uniformly bounded in $L^{p/2}(\Omega, \Lambda^2\mathbb{R}^m \otimes \mathbb{R}^n \otimes \mathbb{R}^n)$, the weak convergence (48) now follows from (49).

Now we claim that, for all $i = 3, \dots, I$,

$$d^{\Gamma_k}\partial_{\gamma_1}u_k \wedge \ldots \wedge d^{\Gamma_k}\partial_{\gamma_i}u_k \rightharpoonup d^{\Gamma}\partial_{\gamma_1}u \wedge \ldots \wedge d^{\Gamma}\partial_{\gamma_i}u \tag{50}$$

weakly in $L^{p/i}(\Omega, \Lambda^i\mathbb{R}^m \otimes \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n)$. It then follows from the convexity of f that

$$F(u) \le \liminf_{k\to\infty} F(u_k).$$

Hence u is a solution of the minimization problem.
We prove (50) by induction on i. The case $i = 2$ has been established above. Now assume that (50) is true for $i-1$ instead of i and write, using (45),

$$\begin{aligned} d^{\Gamma_k}\partial_{\gamma_1}u_k \wedge \ldots \wedge d^{\Gamma_k}\partial_{\gamma_i}u_k &= d^{\Gamma_k}(\partial_{\gamma_1}u_k \otimes d^{\Gamma}\partial_{\gamma_2}u_k \wedge \ldots \wedge d^{\Gamma_k}\partial_{\gamma_i}u_k) \\ &- \frac{1}{2}\partial_{\gamma_1}u_k \otimes R(u_k)(du_k, du_k)\partial_{\gamma_2}u_k \wedge d^{\Gamma_k}\partial_{\gamma_3}u_k \wedge \ldots \wedge d^{\Gamma_k}\partial_{\gamma_i}u_k \\ &+ \ldots \end{aligned} \tag{51}$$

By the inductive hypothesis we have

$$d^{\Gamma_k}\partial_{\gamma_2}u_k \wedge \ldots \wedge d^{\Gamma_k}\partial_{\gamma_i}u_k \rightharpoonup d^{\Gamma}\partial_{\gamma_2}u \wedge \ldots \wedge d^{\Gamma}\partial_{\gamma_i}u$$

weakly in $L^{\frac{p}{i-1}}$. Since (47) implies that $\partial_{\gamma_1}u_k \to \partial_{\gamma_1}u$ strongly in L^q and since

$$\frac{2}{q} + \frac{i-1}{p} \le 1,$$

Hölder's inequality together with (39) implies that

$$d^{\Gamma_k}(\partial_{\gamma_1}u_k \otimes d^{\Gamma_k}\partial_{\gamma_2}u_k \wedge \ldots \wedge d^{\Gamma_k}\partial_{\gamma_i}u_k) \overset{*}{\rightharpoonup} d^{\Gamma}(\partial_{\gamma_1}u \otimes d^{\Gamma}\partial_{\gamma_2}u \wedge \ldots \wedge d^{\Gamma}\partial_{\gamma_i}u)$$

as distributions.
A short calculation shows that

$$\frac{i-2}{p} + \frac{4}{q} \le 1. \tag{52}$$

Hence, by boundedness of R and by dominated convergence, (47) implies

$$\begin{aligned} &\partial_{\gamma_1}u_k \otimes R(u_k)(du_k, du_k)\partial_{\gamma_2}u_k \\ &\qquad \to \partial_{\gamma_1}u \otimes R(u)(du, du)\partial_{\gamma_2}u \text{ strongly in } L^{\left(\frac{p}{i-2}\right)'}. \end{aligned} \tag{53}$$

By the inductive hypothesis, we also know that

$$d^{\Gamma_k}\partial_{\gamma_3}u_k \wedge \ldots \wedge d^{\Gamma_k}\partial_{\gamma_i}u_k \rightharpoonup d^{\Gamma}\partial_{\gamma_3}u \wedge \ldots \wedge d^{\Gamma}\partial_{\gamma_i}u \text{ weakly in } L^{\frac{p}{i-2}}. \tag{54}$$

The convergences (53) and (54) together imply in particular that

$$\begin{aligned} &\partial_{\gamma_1}u_k \otimes R(u_k)(du_k, du_k)\partial_{\gamma_2}u_k \wedge d^{\Gamma_k}\partial_{\gamma_3}u_k \wedge \ldots \wedge d^{\Gamma_k}\partial_{\gamma_i}u_k \\ &\qquad \overset{*}{\rightharpoonup} \partial_{\gamma_1}u \otimes R(u)(du, du)\partial_{\gamma_2}u \wedge d^{\Gamma}\partial_{\gamma_3}u \wedge \ldots \wedge d^{\Gamma}\partial_{\gamma_i}u \end{aligned}$$

weakly-* in the distributional sense. Finally, note that the left-hand side of (51) is uniformly bounded in $L^{\frac{p}{i}}$, simply by Hölder's inequality because $d^{\Gamma}\partial_{\gamma_j}u_k$ is uniformly bounded in L^p for all j, k. $\square$

5 Monotonicity formula

In this section we derive, for general exponents $p > 1$, a monotonicity formula that is analogous to the main monotonicity formula obtained in [6] for the case $p = 2$. Our derivation differs slightly from that in [6], and it corrects two minor computational errors in the lower order terms. Unfortunately, in contrast to the case $p = 2$ studied in [6] and [8], for $p \neq 2$ we are not able to exploit the monotonicity formula in order to derive the Morrey bounds needed to conclude local $W^{2,p}$-integrability. This is due to some difficulties which do not arise in the case $p = 2$, and which do not arise for p-harmonic maps either. They are related to the fact that the derivative of the energy density $|Ddu|^p$ involves the expression $|Ddu|^{p-2}$, which is trivial when $p = 2$ but which does not admit further differentiation when $p \neq 2$.

The Euler-Lagrange equation for the functional

$$\int_\Omega F(|Ddu|^2) \tag{55}$$

is

$$D_\beta D_\alpha \left(F'(|Ddu|^2) D_\alpha \partial_\beta u\right) + F'(|Ddu|^2) R(u)(D_\alpha \partial_\beta u, \partial_\beta u)\partial_\alpha u = 0. \tag{56}$$

We scalar multiply this with $\partial_\gamma u$ to find, after some manipulations, the following stationarity condition for the functional (55):

$$\begin{aligned}\partial_\alpha\partial_\beta \left(F'(|Ddu|^2) D_\beta \partial_\alpha u \cdot \partial_\gamma u\right) - 2\partial_\alpha \left(F'(|Ddu|^2) D_\beta \partial_\gamma u \cdot D_\beta \partial_\alpha u\right)& \\ + \frac{1}{2}\partial_\gamma \left(F(|Ddu|^2)\right) = 0.&\end{aligned} \tag{57}$$

If $F(t) = t^{p/2}$ then $F'(t) = \frac{p}{2t}F(t)$ and so $F'(|Ddu|^2) = \frac{p}{2}|Ddu|^{p-2}$. This leads to the following definition:

Definition 5.1 *A map $u \in W^{2,p}_N(\Omega)$ is said to be stationary for E_p if*

$$\begin{aligned}\frac{p}{2}\partial_\alpha\partial_\beta \left(|Ddu|^{p-2} D_\beta \partial_\alpha u \cdot \partial_\gamma u\right) - p\partial_\alpha \left(|Ddu|^{p-2} D_\beta \partial_\gamma u \cdot D_\beta \partial_\alpha u\right)& \\ + \frac{1}{2}\partial_\gamma \left(|Ddu|^p\right) = 0&\end{aligned} \tag{58}$$

is satisfied in the sense of distributions.

Observe that the left-hand side of (58) is indeed well-defined as a distribution because $|Ddu|^{p-1} \in L^{p'}$. In the case $p = 2$ the formula (58) agrees with the one given in Definition 3.1 in [6].

The monotonicity formula that we derive involves integrals over concentric balls and spheres in Ω. For simplicity, we assume that $0 \in \Omega$, so that we can consider balls B_r centered at 0. Let ∂_r denote the radial derivative with respect to this center and D_r the covariant derivative in radial direction, that is, we define

$$\partial_r u = \frac{x_\alpha}{|x|}\partial_\alpha u \text{ and } D_r X = \frac{x_\alpha}{|x|} D_\alpha X.$$

Furthermore, let $\mathcal{H}^{m-1}$ denote the $(m-1)$-dimensional Hausdorff measure. We use the abbreviation $\tau(u) = D_\alpha \partial_\alpha u$.

Proposition 5.2 *Let $p > 1$ and assume that $u \in W^{2,p}_N(B_1)$ is stationary for E_p. Then the function*

$$\mathcal{E}(\rho) = \frac{\rho^{2p-m}}{2}\int_{B_\rho} |Ddu|^p + \frac{p}{4}\rho^{2p-m}\int_{\partial B_\rho} |Ddu|^{p-2}\partial_r|\partial_r u|^2 \; d\mathcal{H}^{m-1}(x)$$
$$-\frac{p}{2}\int_{B_\rho}\left\{\partial_r|du|^2 + \tau(u)\cdot\partial_r u + \frac{2p-m-2}{2}\partial_r|\partial_r u|^2\right\}|x|^{2p-m-1}|Ddu|^{p-2}$$

satisfies the following monotonicity formula:

$$\mathcal{E}(R) - \mathcal{E}(r) = p\int_{B_R\setminus B_r} |x|^{2p-m}|Ddu|^{p-2}|D_r du|^2. \tag{59}$$

Proof. Let $R \in (0,1)$. In order to derive a monotonicity formula, we test (57) with $\psi_R(|x|)x_\gamma$, where $\psi_R(t) = \psi(t/R)$, where $\psi \in C_0^\infty(B_1)$ is fixed. We find:

$$0 = \int F'(|Ddu|^2)D_\beta\partial_\alpha u\cdot\partial_\gamma u\Bigg[\partial_\alpha\partial_\beta(\psi_R(|x|))x_\gamma$$
$$+ \partial_\beta(\psi_R(|x|))\delta_{\alpha\gamma} + \partial_\alpha(\psi_R(|x|))\delta_{\beta\gamma}\Bigg]$$
$$+ \int 2F'(|Ddu|^2)D_\beta\partial_\gamma u\cdot D_\beta\partial_\alpha u\Bigg[\delta_{\alpha\gamma}\psi_R(|x|) + x_\gamma\partial_\alpha(\psi_R(|x|))\Bigg]$$
$$- \int \frac{F(|Ddu|^2)}{2}\Bigg[m\psi_R(|x|) + x_\gamma\partial_\gamma(\psi_R(|x|))\Bigg].$$

Using
$$\nabla(\psi_R(|x|)) = \frac{x}{|x|}\psi_R'(|x|)$$
and
$$\nabla^2(\psi_R(|x|)) = \frac{1}{|x|}(I - \frac{x}{|x|}\otimes\frac{x}{|x|})\psi_R'(|x|) + \frac{x}{|x|}\otimes\frac{x}{|x|}\psi_R''(|x|),$$
and introducing
$$a(x) = 2|x|F'(|Ddu|^2)|D_r du|^2$$
$$+ F'(|Ddu|^2)\left(\partial_r|du|^2 + \tau(u)\cdot\partial_r u - \frac{1}{2}\partial_r|\partial_r u|^2\right),$$
and
$$q(x) = a(x) - \frac{|x|}{2}F(|Ddu|^2)$$
as well as
$$b(x) = \frac{x_\alpha x_\beta x_\gamma}{|x|^2}F'(|Ddu|^2)D_\beta\partial_\alpha u\cdot\partial_\gamma u = \frac{|x|}{2}F'(|Ddu|^2)\partial_r|\partial_r u|^2,$$

the above equality can be written as

$$0 = \frac{2p-m}{2} \int F(|Ddu(x)|^2)\psi_R(|x|)\ dx \\ + \int q(x)\psi_R'(|x|) + b(x)\psi_R''(|x|)\ dx. \tag{60}$$

In order to obtain the first term we have used that $F(t) = t^{p/2}$, so that

$$|Ddu|^2 F'(|Ddu|^2) = \frac{p}{2} F(|Ddu|^2).$$

Define

$$\mathcal{G}(R) = \frac{1}{2} R^{2p-m} \int F(|Ddu|^2)\psi_R.$$

Since $\frac{d}{dR}\psi_R(|x|) = -\frac{|x|}{R^2}\psi'(|x|/R)$, we clearly have

$$\frac{d}{dR}\mathcal{G}(R) = \frac{2p-m}{2} R^{2p-m-1} \int F(|Ddu|^2)\psi_R \\ - \frac{R^{2p-m-2}}{2} \int |x| F(|Ddu|^2)\psi'\left(\frac{|x|}{R}\right).$$

Using (60) to replace the first term on the right, we find

$$\begin{aligned} \frac{d}{dR}\mathcal{G}(R) &= -R^{2p-m-2} \int \left(\frac{|x|}{2} F(|Ddu|^2) + q(x)\right) \psi'\left(\frac{|x|}{R}\right) \\ &\quad - R^{2p-m-3} \int b(x)\psi''\left(\frac{|x|}{R}\right) \\ &= -R^{2p-m-2} \int a(x)\psi'\left(\frac{|x|}{R}\right) - R^{2p-m-3} \int b(x)\psi''\left(\frac{|x|}{R}\right). \end{aligned} \tag{61}$$

This agrees with (the derivative of) a formula on page 1665 in [6] if we formally take $n = m + 4 - 2p$ there. With this change we can argue exactly as in [6] to deduce from (61) the formula

$$\mathcal{G}(R) - \mathcal{G}(r) = \left[\rho^{2p-m-1} \int \frac{b(x)}{|x|}\psi'\left(\frac{|x|}{\rho}\right)\right]_{\rho=r}^{R} \\ + \int \left(\frac{a(x)}{|x|} + (2p-m-1)\frac{b(x)}{|x|^2}\right) g(r,R,x), \tag{62}$$

where

$$g(r,R,x) = \left[\rho^{2p-m}\psi\left(\frac{|x|}{\rho}\right)\right]_r^R + (m-2p)\int_r^R \rho^{2p-m-1}\psi\left(\frac{|x|}{\rho}\right)\ d\rho.$$

Next we replace ψ by a sequence of functions $\psi_k \in C^\infty(\mathbb{R})$ such that $\psi_k' \leq 0$ and

$$\psi_k = \begin{cases} 1 & \text{in } \left(-\infty, 1-\frac{1}{k}\right) \\ 0 & \text{in } [1,\infty) \end{cases}$$

for all $k \in \mathbb{N}$. In particular, ψ_k are uniformly bounded by 1 and converge in measure to the characteristic function of the set $(-\infty, 1)$. Then (with obvious notation)

$$\mathcal{G}_k(\rho) \to \frac{\rho^{2p-m}}{2} \int_{B_\rho} F(|Ddu|^2)$$

and, by the coarea formula,

$$\rho^{2p-m-1} \int \frac{b(x)}{|x|} \psi_k' \left(\frac{|x|}{\rho} \right) \to -\rho^{2p-m-1} \int_{\partial B_\rho} b(x) \ d\mathcal{H}^{m-1}(x).$$

We thus conclude that

$$\begin{aligned} &\mathcal{G}_k(\rho) - \rho^{2p-m-1} \int \frac{b(x)}{|x|} \psi_k' \left(\frac{|x|}{\rho} \right) \\ &\qquad \to \frac{\rho^{2p-m}}{2} \int_{B_\rho} F(|Ddu|^2) + \rho^{2p-m-1} \int_{\partial B_\rho} b(x) \ d\mathcal{H}^{m-1}(x) \end{aligned}$$

as ψ_k converges to the characteristic function of $(-\infty, 1)$. By (62) this implies

$$\begin{aligned} &\Big[\frac{\rho^{2p-m}}{2} \int_{B_\rho} F(|Ddu|^2) + \rho^{2p-m-1} \int_{\partial B_\rho} b(x) \ d\mathcal{H}^{m-1}(x) \Big]_{\rho=r}^{R} \\ &\qquad = \int_{B_R \setminus B_r} \left(a(x) + (2p - m - 1) \frac{b(x)}{|x|} \right) |x|^{2p-m-1}, \quad (63) \end{aligned}$$

because

$$g_k(r, R, \cdot) \to \chi_{B_R \setminus B_r} |\cdot|^{2p-m}$$

in measure and uniformly bounded by a constant.
Now we insert back the definitions of a and b into (63). This gives:

$$\begin{aligned} &\Big[\frac{\rho^{2p-m}}{2} \int_{B_\rho} F(|Ddu|^2) + \rho^{2p-m-1} \int_{\partial B_\rho} \frac{|x|}{2} F'(|Ddu|^2) \partial_r |\partial_r u|^2 \ d\mathcal{H}^{m-1}(x) \Big]_{\rho=r}^{R} \\ &= \int_{B_R \setminus B_r} \Bigg\{ 2|x| F'(|Ddu|^2) |D_r du|^2 \\ &+ F'(|Ddu|^2) \left(\partial_r |du|^2 + \tau(u) \cdot \partial_r u - \frac{1}{2} \partial_r |\partial_r u|^2 \right) \\ &+ \frac{2p - m - 1}{2} F'(|Ddu|^2) \partial_r |\partial_r u|^2 \Bigg\} |x|^{2p-m-1} \\ &= \int_{B_R \setminus B_r} 2|x|^{2p-m} F'(|Ddu|^2) |D_r du|^2 \\ &+ \Bigg\{ F'(|Ddu|^2) \left(\partial_r |du|^2 + \tau(u) \cdot \partial_r u - \frac{1}{2} \partial_r |\partial_r u|^2 \right) \\ &+ \frac{2p - m - 1}{2} F'(|Ddu|^2) \partial_r |\partial_r u|^2 \Bigg\} |x|^{2p-m-1} \end{aligned}$$

Writing

$$
\begin{aligned}
\mathcal{E}(\rho) &= \frac{\rho^{2p-m}}{2}\int_{B_\rho} F(|Ddu|^2) + \rho^{2p-m}\int_{\partial B_\rho} \frac{1}{2}F'(|Ddu|^2)\partial_r|\partial_r u|^2 \; d\mathcal{H}^{m-1}(x) \\
&\quad - \int_{B_\rho} \left\{ \partial_r|du|^2 + \tau(u)\cdot\partial_r u - \frac{1}{2}\partial_r|\partial_r u|^2 \right. \\
&\quad \left. + \frac{2p-m-1}{2}\partial_r|\partial_r u|^2 \right\} |x|^{2p-m-1}F'(|Ddu|^2),
\end{aligned}
$$

the above equality can be concisely written as the following monotonicity formula for $\mathcal{E}$:

$$
\mathcal{E}(R) - \mathcal{E}(r) = 2\int_{B_R\setminus B_r} |x|^{2p-m}F'(|Ddu|^2)|D_r du|^2.
$$

Inserting $F(t) = t^{p/2}$, we deduce the claim. □

In the following corollary we derive the monotonicity formula from [6] by somewhat different arguments. We present it in some detail, because the formula in [6] contains two minor errors in the boundary integral.

Corollary 5.3 *Assume that $u \in W^{2,2}_N(B_1)$ is stationary for E_2. Then the function*

$$
\begin{aligned}
\mathcal{F}(\rho) &= \frac{\rho^{4-m}}{2}\int_{B_\rho} |Ddu|^2 \\
&\quad + \frac{\rho^{3-m}}{2}\int_{\partial B_\rho} \Big(3|du|^2 + (m-4)|\partial_r u|^2 + \rho\partial_r\big(|\partial_r u|^2\big) \Big) \; d\mathcal{H}^{m-1}
\end{aligned}
$$

satisfies the monotonicity formula

$$
\begin{aligned}
\mathcal{F}(R) - \mathcal{F}(r) = 2\int_{B_R\setminus B_r} & |x|^{2-m}|D\big(|x|\partial_r u\big)|^2 \\
& + 2(m-2)\int_{B_R\setminus B_r} |\partial_r u|^2|x|^{2-m} \; dx.
\end{aligned} \tag{64}
$$

Proof. Using

$$
\nabla\frac{x}{|x|} = \frac{1}{|x|}\left(I - \frac{x\otimes x}{|x|^2} \right),
$$

it is easy to check the general formula (for regular enough $f : B_R \to \mathbb{R}$)

$$
\int_{B_R\setminus B_r} (\partial_r f)(x)dx = (1-m)\int_{B_R\setminus B_r} \frac{f(x)}{|x|} \; dx + \left[\int_{\partial B_\rho} f \; d\mathcal{H}^{m-1} \right]_r^R. \tag{65}
$$

This, in turn, readily implies the formula (for regular enough $\mu : \mathbb{R} \to \mathbb{R}$)

$$
\begin{aligned}
\int_{B_R\setminus B_r} (\partial_r f)(x)\; \mu(|x|) \; dx = -\int_{B_R\setminus B_r} & f(x)\Big(\mu'(|x|) + (m-1)\frac{\mu(|x|)}{|x|} \Big) \; dx \\
& + \Big[\mu(\rho)\int_{\partial B_\rho} f \; d\mathcal{H}^{m-1} \Big]_r^R.
\end{aligned} \tag{66}
$$

Now we apply formula (66) with $f = |du|^2$ and with $\mu(t) = t^{3-m}$. This gives

$$\begin{aligned}\int_{B_R\setminus B_r} \partial_r |du|^2 |x|^{3-m} &= -2\int_{B_R\setminus B_r} |du|^2 |x|^{2-m}\, dx \\ &\quad + \Big[\rho^{3-m}\int_{\partial B_\rho} |du|^2\, d\mathcal{H}^{m-1}\Big]_r^R.\end{aligned}\tag{67}$$

Now we apply formula (66) with $f = |\partial_r u|^2$ and with $\mu(t) = t^{3-m}$. This gives

$$\begin{aligned}\int_{B_R\setminus B_r} \partial_r |\partial_r u|^2 |x|^{3-m} &= -2\int_{B_R\setminus B_r} |\partial_r u|^2 |x|^{2-m}\, dx \\ &\quad + \Big[\rho^{3-m}\int_{\partial B_\rho} |\partial_r u|^2\, d\mathcal{H}^{m-1}\Big]_r^R.\end{aligned}\tag{68}$$

As in [6] we compute:

$$\begin{aligned}\int_{B_R\setminus B_r} (\tau(u)\cdot\partial_r u)|x|^{3-m} &= -\frac{1}{2}\int_{B_R\setminus B_r} \partial_r |du|^2 |x|^{3-m} - \int_{B_R\setminus B_r} |du|^2|x|^{2-m} \\ &\quad + (m-2)\int |\partial_r u|^2 |x|^{2-m} + \Big[\rho^{3-m}\int_{\partial B_\rho} |\partial_r u|^2\Big]_{\rho=r}^R.\end{aligned}$$

Using (67) we therefore conclude:

$$\begin{aligned}\int_{B_R\setminus B_r} (\tau(u)\cdot\partial_r u)|x|^{3-m} &= (m-2)\int |\partial_r u|^2|x|^{2-m} \\ &\quad + \Big[\rho^{3-m}\int_{\partial B_\rho} |\partial_r u|^2 - \frac{|du|^2}{2}\, d\mathcal{H}^{m-1}\Big]_{\rho=r}^R.\end{aligned}\tag{69}$$

Plugging (67), (68), (69) into

$$\begin{aligned}\mathcal{E}_2(\rho) &= \frac{\rho^{4-m}}{2}\int_{B_\rho} |Ddu|^2 + \frac{1}{2}\rho^{3-m}\int_{\partial B_\rho} |x|\partial_r|\partial_r u|^2\, d\mathcal{H}^{m-1}(x) \\ &\quad - \int_{B_\rho}\Big\{\partial_r|du|^2 + \tau(u)\cdot\partial_r u + \frac{2-m}{2}\partial_r|\partial_r u|^2\Big\}|x|^{3-m},\end{aligned}$$

and using (59), we find:

$$\begin{aligned}&2\int_{B_R\setminus B_r} |x|^{4-m}|D_r du|^2 \\ &= \Big[\frac{\rho^{4-m}}{2}\int_{B_\rho}|Ddu|^2 + \frac{1}{2}\rho^{3-m}\int_{\partial B_\rho}|x|\partial_r|\partial_r u|^2\, d\mathcal{H}^{m-1}(x)\Big]_r^R \\ &- 2\int_{B_R\setminus B_r}\Big(-|du|^2 + (m-2)|\partial_r u|^2\Big)|x|^{2-m}\, dx \\ &- \Big[\rho^{3-m}\int_{\partial B_\rho}\Big(\frac{|du|^2}{2} + \frac{4-m}{2}|\partial_r u|^2\Big)\, d\mathcal{H}^{m-1}\Big]_r^R.\end{aligned}\tag{70}$$

Now note the simple equality

$$|x|^2|D_r du|^2 = |D(|x|\partial_r u)|^2 - |du|^2 - |x|\partial_r|du|^2.\tag{71}$$

Hence using (67) we find

$$
\begin{aligned}
&\int_{B_R\setminus B_r} |x|^{4-m}|D_r du|^2 \\
&\qquad = \int_{B_R\setminus B_r} |x|^{2-m}|D(|x|\partial_r u)|^2 - \int_{B_R\setminus B_r} |x|^{2-m}|du|^2 \\
&\qquad - \int_{B_R\setminus B_r} |x|^{3-m}\partial_r|du|^2 \\
&\qquad = \int_{B_R\setminus B_r} |x|^{2-m}|D(|x|\partial_r u)|^2 - \int_{B_R\setminus B_r} |x|^{2-m}|du|^2 \\
&\qquad + 2\int_{B_R\setminus B_r} |du|^2|x|^{2-m}\, dx - \Big[\rho^{3-m}\int_{\partial B_\rho} |du|^2 \, d\mathcal{H}^{m-1}\Big]_r^R .
\end{aligned}
$$

We plug this into the left-hand side of (70) to conclude, after some easy simplifications,

$$
\begin{aligned}
2\int_{B_R\setminus B_r} |x|^{2-m}|D(|x|\partial_r u)|^2 &= \left[\frac{\rho^{4-m}}{2}\int_{B_\rho} |Ddu|^2\right]_r^R \\
&- 2(m-2)\int_{B_R\setminus B_r} |\partial_r u|^2|x|^{2-m}\, dx \\
&+ \Big[\rho^{3-m}\int_{\partial B_\rho} \Big(\frac{|x|}{2}\partial_r|\partial_r u|^2 + \frac{3}{2}|du|^2 + \frac{m-4}{2}|\partial_r u|^2\Big)\, d\mathcal{H}^{m-1}\Big]_r^R .
\end{aligned}
$$

This is (64). □

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